one writes as the function to be minimized, for a unit load on the $i$th coordinate,

$$\varphi = \frac{1}{2}\sum_{\alpha,\beta} b_{\alpha,\beta}k_\alpha k_\beta + \sum_{\alpha,\beta} \lambda_\alpha(V_{\alpha,\beta}k_\beta - \delta_\alpha{}^i) \quad (6)$$

where $\lambda\alpha$ are Lagrange multipliers and $\delta_\alpha{}^i$ is the Kroneker delta. Setting the partials with respect to the $k_\gamma$ and the $\lambda_\gamma$ to zero gives, on noting that $B$ is symmetrical,

$$\frac{\partial\varphi}{\partial k_\gamma} = \sum_{\alpha=1}^{r} b_{\alpha,\gamma}k_\alpha + \sum_\alpha \lambda_\alpha V_{\alpha\gamma} = 0 \qquad \gamma = 1, 2, \ldots r \quad (7)$$

$$\frac{\partial\varphi}{\partial\lambda_\gamma} = \sum_\beta V_{\gamma\beta}k_\beta - \delta_\gamma{}^i = 0 \quad (8)$$

Returning to matrix notation and letting $\Gamma = \mathrm{col}(\lambda_1,\lambda_2, \ldots\lambda_r)$, transforms Eqs. (7) and (8) into

$$Bk^{(i)} + V^T\Gamma^{(i)} = 0 \quad (9)$$

$$e_i = Vk^{(i)} \quad (10)$$

where $e_i$ is a column vector with a one in the $i$th place and zeros elsewhere. The superscript $i$ has been added to the $k$ and $\Gamma$ to indicate that here the load $p = e_i$. Assuming that $B$ is nonsingular, one now premultiplies (9) by $B^{-1}$ and transposes, obtaining

$$k^{(i)} = -B^{-1}V^T\Gamma^{(i)} \quad (11)$$

Substituting this into Eq. (10) and premultiplying the result by $(VB^{-1}V^T)^{-1}$ and substituting $\Gamma^{(i)}$ into (11) gives

$$k^{(i)} = B^{-1}V^T(VB^{-1}V^T)^{-1}e_i = Me_i \quad (12)$$

By Ref. 1 the flexibility matrix of the supported element is

$$F_{ij} = \int_V \sigma_{(i)}{}^T\epsilon_{(j)}dV \quad (13)$$

where $\sigma_{(i)}$ is the stress vector at a point in the element caused by a unit load at coordinate $i$, and $\epsilon_{(j)} = \mathrm{col}(\epsilon_x{}^j,\epsilon_y{}^j,\epsilon_z{}^j,\gamma_{xy}{}^j, \gamma_{yz}{}^j,\gamma_{zx}{}^j)$ is the strain vector caused by unit load at $j$, the integration being taken over the volume $V$ of the element. Expressing the customary relation between stress and strain in matrix notation as

$$\epsilon = N\sigma \quad (14)$$

and using Eqs. (1) and (12) converts Eq. (13) into

$$F_{ij} = e_i{}^TM^TGMe_j \quad \text{where} \quad G = \int_V U^TNUdV \quad (15)$$

This amounts to

$$F = M^TGM \quad (16)$$

By inversion of Eq. (16) one obtains the $m \times m$ stiffness matrix of the supported element:

$$\bar{S} = F^{-1} \quad (17)$$

In order to obtain the $n \times n$ stiffness matrix $S$ of the unsupported element, the transformation matrix $H$ is defined satisfying

$$\begin{pmatrix} p \\ p_F \end{pmatrix} = Hp \quad (18)$$

where $p_F$ = support loads determined from equilibrium equations as linear combinations of the applied loads so that

$$S = H\bar{S}H^T \quad (19)$$

In order to show the nonsingularity of the matrices just inverted, Eq. (3) can be written as

$$\bar{W} = \frac{1}{2}\oint_S (\sigma_n{}^2 + \tau_{n1}{}^2 + \tau_{n2}{}^2)dS \quad (20)$$

which will be greater than zero unless $\sigma_n = \tau_{n1} = \tau_{n2} = 0$ everywhere, in which case $k = 0$. This amounts to saying that $B$ is positive definite in Eq. (5) and hence nonsingular. Again, $V$ in Eq. (2) should be of rank $m$, i.e., the loads should be independent, and therefore $V\,B^{-1}V^T$ is also of rank $m$ and consequently nonsingular. For a unit load on the $i$th coordinate, the corresponding $k^{(i)}$ in Eq. (12) will not be zero, since $M$ has rank $m$. The strain energy stored in the element is

$$\bar{U} = \frac{1}{2}\oint_V \epsilon_{(i)}{}^T\sigma_{(i)}dV = k^{(i)T}Gk^{(i)} \quad (21)$$

This can be zero only if the load is zero; hence $k = 0$. In other words $G$ is positive definite and hence nonsingular.

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# Support Interference Effects on the Supersonic Wake


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As it is generally recognized that the presence of a sting, no matter how feasibly small, is likely to affect base pressure and base heating, the use of side-mounted or wire-supported models is usually relied upon in order to obtain essentially disturbance-free measurements in the base region. During the development of the free-flight testing technique in the Jet Propulsion Laboratory continuous wind tunnels,[1] it was observed that the small diameter wires (used to support the models prior to their free-flight trajectories) usually had a significant effect upon the shape of the wake separation region. The purpose of this article is to demonstrate that the use of either side-mounts or wires to support a model in order to obtain interference-free base region measurements is not necessarily an adequate approach.

Concurrent with the acquisition of free-flight model-wake spark-chlieren pictures,[2] interference effect of small diameter-wire supports (0 to 3% of model diameter) upon the separation region shape was investigated for spheres and various cone models through the Mach number $M$ range from $M$ = 1.3 to $M$ = 5 and for several cones at $M$ = 9. The presence of a single traverse vertical wire support did alter noticeably the sphere separation region shape at $1.3 < M < 5$. Spark schlieren pictures at $M$ = 3 in Fig. 1a indicate this typical interference effect. As the diameter of the vertical wire was increased from zero (free-flight) to 0.040 in., the position of the wake neck moved toward the sphere. Figure 1b is a graphical presentation of this wake interference phenomenon at several Reynolds numbers; the definition of the characteristic wake length, $L$, being shown in Fig. 1c. From these results it appears that any wire capable of supporting a sphere will alter the wake.

In Fig. 1a the schlieren pictures indicate that the flow field in the plane of the wire support has no obvious major dis-


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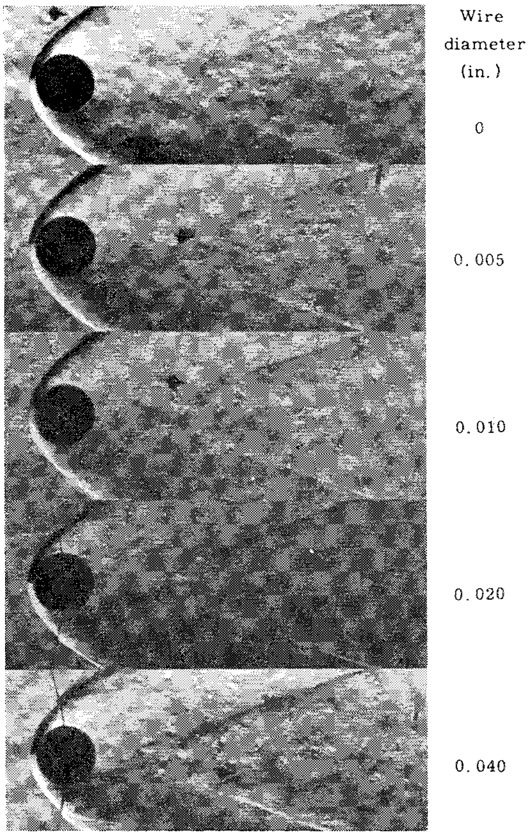
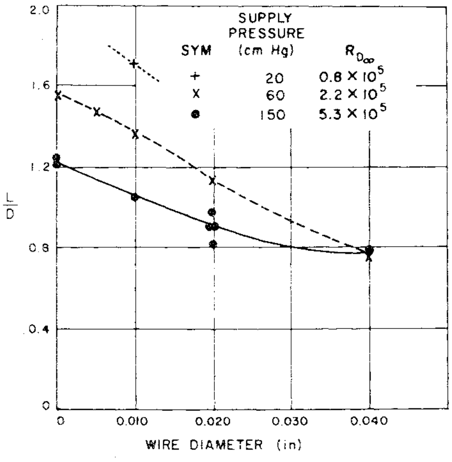
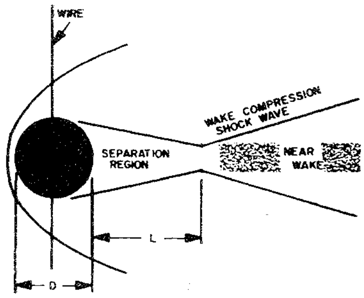



a)




b)




c)

Fig. 1 Effect of diameter of vertical wire support on sphere wakes; $M = 3$, supply pressure = 60 cm Hg, $R_{D\infty} = 2.2 \times 10^5$, model diameter = $1\frac{1}{2}$ in.

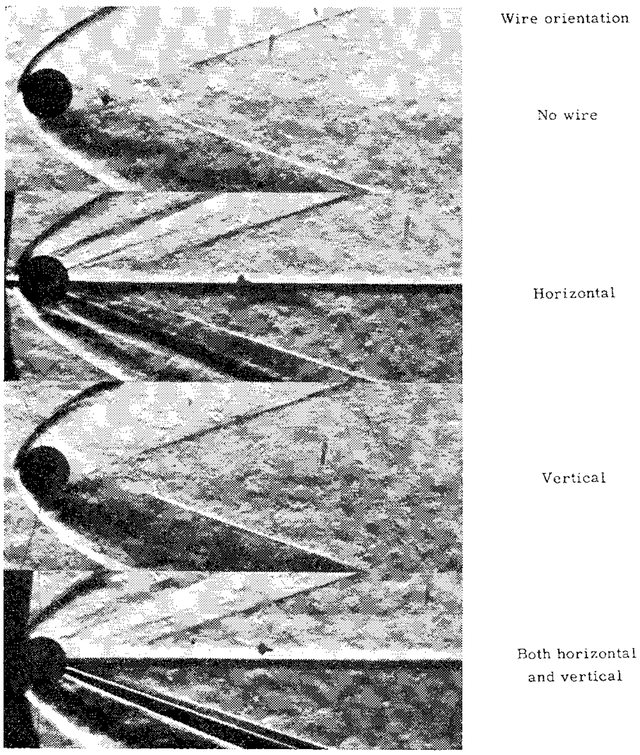
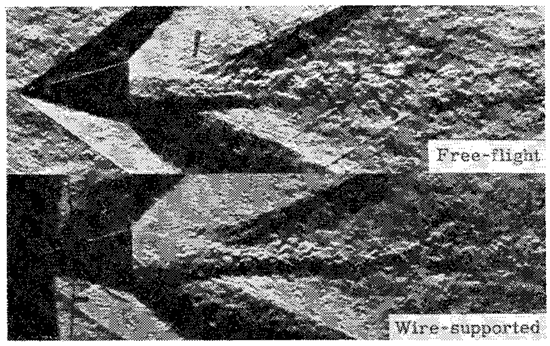
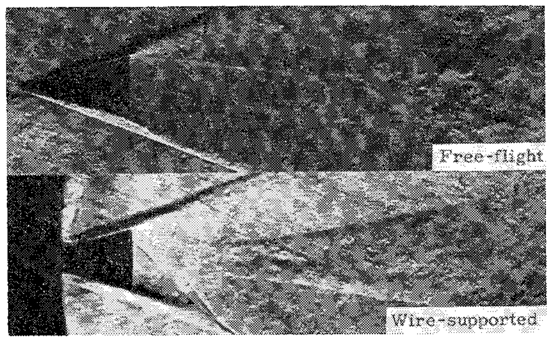



Fig. 2 Effect of support wire orientation on sphere wakes; $M = 3$, supply pressure = 150 cm Hg, $R_{D\infty} = 5.3 \times 10^5$, model diameter = $1\frac{1}{2}$ in., wire diameter = 0.020 in.




a) $M = 2.0$, supply pressure = 140 cm Hg, $R_{D\infty} = 8 \times 10$




b) $M = 4.7$, supply pressure = 330 cm Hg, $R_{D\infty} = 4.5 \times 10^5$

Fig. 3 Effect of wire support on cone wakes; 30° apex-angle, model diameter = $1\frac{1}{2}$ in., wire diameter = 0.026 in.

turbance due to the wire. As can be seen in Fig. 2, the flow field in the plane normal to the wire support is severely disturbed by the presence of the wire. Hence, it is not surprising that a seemingly insignificant wire support can materially alter the wake shape.

A similar interference investigation was carried out for 30° included-angle cones at $1.6 < M < 4.7$. At $M < 2$, there did not appear to be a strong effect of the support wire (about 2% of the cone diameter) on the shape of the separation region for the high Reynolds number condition (Fig. 3a). But at the lower Reynolds numbers ($<2 \times 10^5$), when the wake was definitely laminar to beyond the neck region, the presence of the wire support did decrease the distance of the wake neck to the model. However, at $M > 2$, the effect of the wire support was quite pronounced at high as well as at low Reynolds numbers; a typical example at $M = 4.7$ being shown in Fig. 3b. As in the case of the sphere, the presence of the wire support moved the wake neck toward the cone base. Investigations at $M = 9$ in the Jet Propulsion Laboratory 21-in. hypersonic wind tunnel show that the normally convergent wakes of free-flight cone models become divergent when the $1\frac{1}{2}$-in.-diam models are supported on a single traverse 0.024-in.-diam wire.

Sketchy experimental evidence indicates that when the model boundary layer is turbulent, the presence of a wire support does not materially alter the shape of the wake separation region. This has been observed at $M = 4$ for blunted cones of nose radius to base radius ratios of 0.05 and 0.50. At $M = 1.6$ and $M = 2$, when the wire support did not alter the wake separation region of the 30° apex-angle cone, the shape of the separation region of these models was the same whether or not the model boundary layer was laminar or turbulent (obtained by tripping the boundary layer).

A further word of caution pertains to outside disturbances in the wake which may propagate forward to the model base region. Even for free-flight models, an object in the wake or a shock-wave (reflected model bow shock or from some other source such as a sabot) intersecting the wake, not far enough downstream of the model, may alter the base region flow field. Drag coefficients of spheres were measured during the work described in Ref. 1. At $M = 1.3$ when a $1\frac{1}{2}$-in.-diam sphere was used in the Jet Propulsion Laboratory 20-in. supersonic wind tunnel, the bow shock intersected the wake about 2 diam downstream from the base. This interference caused a 4% decrease in the measured drag coefficient relative to ballistic range data. When the bow shock intersected the wake five or more diameters from the base, the measured drag coefficients were within 1% of ballistic range data. A similar decrease in the drag coefficient has been observed on sting-mounted spheres (Ref. 3, Appendix) when the bow shock wave intersects the wake too near the model.

From the support-wire wake-interference studies, it has been shown that such a manner of supporting models is likely to cause incorrect base flow conditions. Even if it is suspected that a wire support, or some other seemingly negligible support, will not alter the model base flow under the particular conditions being investigated, such should be demonstrated to be the case and not just be assumed. Also, any outside disturbance in the wake must be kept far enough downstream in order that it does not propagate forward to the model and alter the base region flow field.

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# Scatter of Observed Buckling Loads of Pressurized Shells

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INDIVIDUAL experimenters who have determined the buckling load of shells with external pressure generally have shown self-consistent results, but when a large number of experiments are compared it is difficult to discern even trends in the data. Evidence of the state of affairs can be found in the graphical displays of experimental data given in Refs. 1 and 2. Reference 1 also contains experimental data that indicate the general validity of the Biezeno solution for the critical load of a spherical cap compressed by a concentrated force. The difference in the correlation between theory and experiment in the two cases is marked, even to the untrained eye. Recent experiments reported in Ref. 3 show that the classical buckling load can be reached very nearly on axially loaded cylinders. However, according to Refs. 3 and 4, lower buckling loads can be obtained by using shells with artificially induced imperfections.

The presence of minute imperfections in shells has been postulated to explain the discrepancy between theoretical and experimental results, and nonlinear theory has helped to clarify the situation, since it now is known that the instability is of the *transitional* rather than of the *bifurcation* type. That is, the load deformation curve is of the general shape shown in Fig. 1. The supposed presence of vanishingly small imperfections points to a buckling load of the magnitude indicated by $A$ or $C$ of Fig. 1 but does not explain fully how the shell equilibrium configuration can jump from $A$ to $C$. Furthermore, imperfections (if actually present) do not seem to affect shells loaded by concentrated forces, for the agreement between theory and observation is good there, as previously mentioned.

These considerations led the writer to speculate on other possible explanations of the wide scatter of experimental values. One idea that appears to hold some promise is the following: *a mechanical system that is loaded by nonconservative generalized forces may buckle dynamically as well as statically.*

The concept of dynamic instability of compressed columns was introduced by Max Beck,[5] who showed that a finite dynamic buckling load existed in the case of a cantilever column loaded tangentially at its unsupported end, even though no static buckling load could be found. The critical load for this type of buckling appropriately may be called a "Beck load" $P_B$ in contrast to the more usual "Euler load" $P_E$ that governs bifurcation-type buckling failure.

The concepts involved here are understood readily by the analysis of a simple linked bar system as illustrated in Fig. 2. Two massless bars are linked together and carry point masses of magnitude $m$ at the joint and at the free end. The bars are restrained from free rotation by torsional springs, each of modulus $k$ as shown. The "column" thus formed is compressed by two forces, $P$ that remains directed along the upper link and $Q$ that remains vertical. Equations of motion are

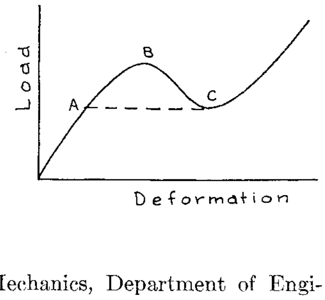


**Fig. 1 Shape of load deformation curve.**

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